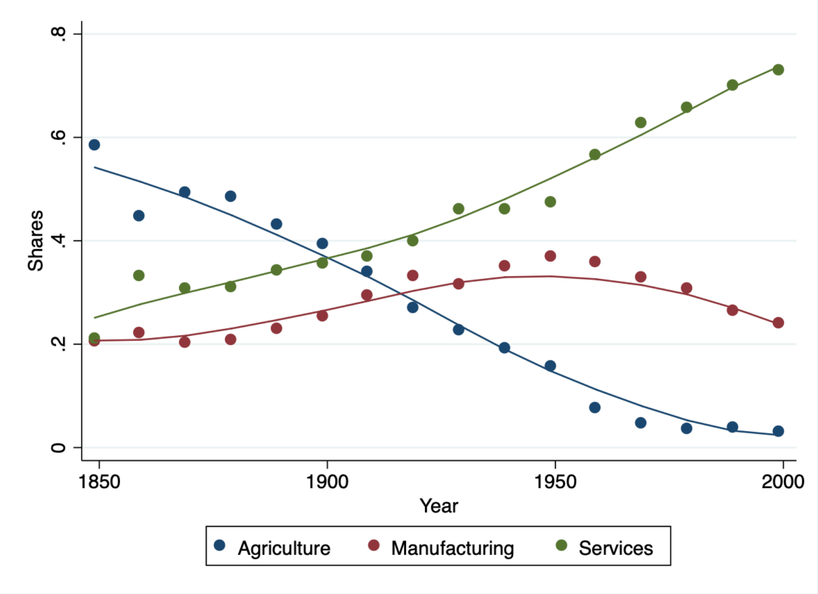
Structural Change in Innovation[1]

Diego Comin (Dartmouth)
Danial Lashkari (NYFed)
Martí Mestieri (IAE-CSIC)

[1] The views of this paper solely represents those of the authors and not necessarily those of the NYFed or the Federal Reserve System.

Structural Change in US Production (Employment Sh.)



Document Structural Change in Innovation and Model it

Document there is Structural Change in Innovation and TFP

- Look across 3 broad sectors (ag,ind, serv).
- Patents: Historical US Data since 1856.
  - Twentieth century: UK, Germany, France and Japan
- R&D: US Fixed asset tables
- TFP growth: US TFP since 1947.

Growth Model w/ Endogenous Direction of Innovation

- Direction of innovation decided by private firms, depends on:
  1. Market size: modulated by nonhomotheticicities.
  2. Sectoral differences in innovating technology.

⇒ Sectoral TFP growth: endogenous outcome of directed innov.

- Integrated framework for two canonical views struct. change:
  - Technological Progress and Nonhomotheticity.

Results: CGP Characterization and Dynamics

- Model generates demand pull and technology push.
- Characterize analitically the asymptotic behavior of model.
- Two quantification exercises:
  1. Constant Growth Path + Linearization $\sim$ (very) long-run
  2. Transitional dynamics away from CGP
- Model Quantification Strategy
  - Use patent data to estimate model's innovation parameters.
- Findings:
  1. Looking backwards: transitional dynamics replicate joint evolution of sectoral shares and TFP growth.
  2. Looking forward: future TFP growth is slowing down.

Evidence

Use Patents to Investigate Long-run Innovation Patterns

- Patents provide a traceable measure of innovation.
- Two sources:
  1. CUSP (Berkes, 2019): Universe of US patents (1836-2010) and their technology category.
  2. PatStat: Patents filed in major Western economies.

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- Pattern:

  Agriculture & Heating (-1876) →
  Engineering Elements (1877-1958) →
  Chemistry (60s) → Measuring (70s) →
  Medicine (80s) → Computing(since mid 90s)

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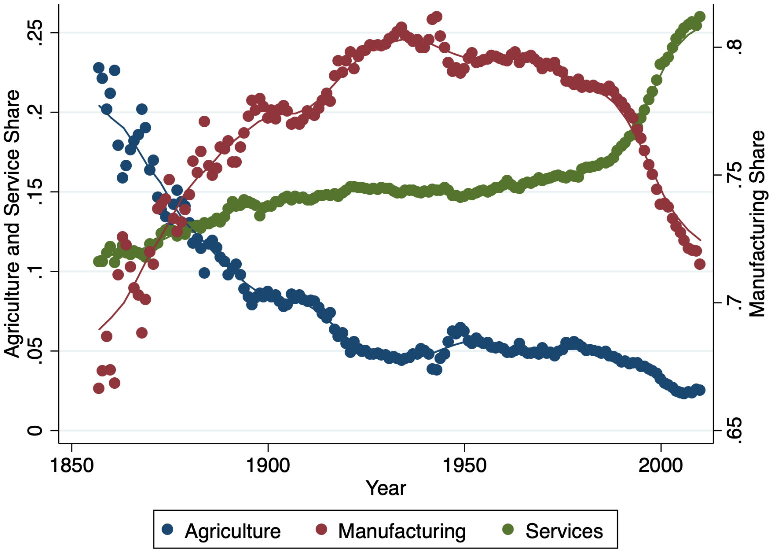
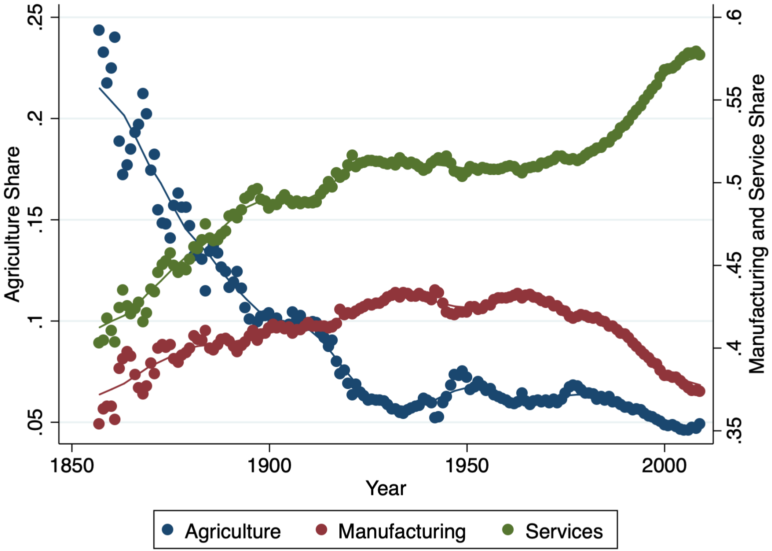
  Chemistry (60s) $\rightarrow$ Measuring (70s) $\rightarrow$
  Medicine (80s) $\rightarrow$ Computing(since mid 90s)
- Classify Patent Tech. Category into Agri., Manu. or Serv.
    - Baseline: x-walk according to assignee.
    - Robust to x-walk based on industry of use.

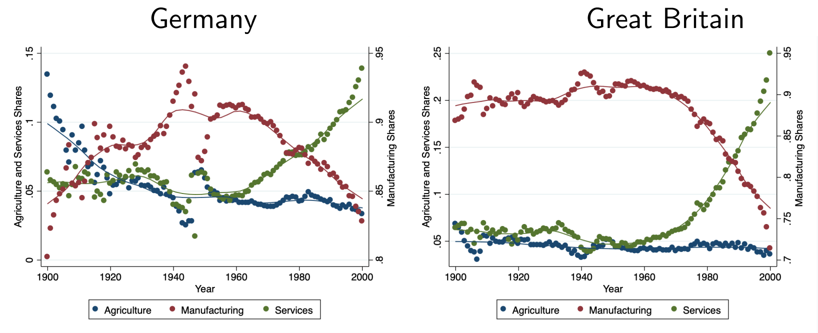
Structural Change in US Patenting



Structural Change in US Patenting, NLP x-walk



Structural Change in German and British Patenting



- Similar for France (1900-), and Japan (1970-).
- Similar with income per capita and OECD countries.

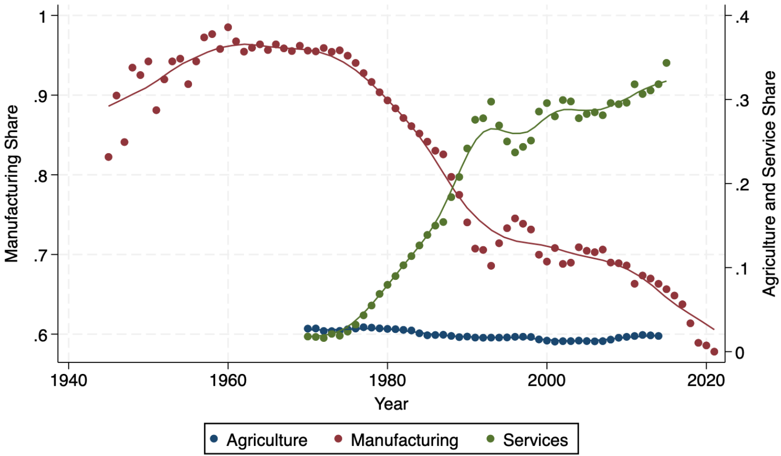
Citation Patterns: Standing on the Shoulders of Giants

Table: Raw Citation Probabilities, US 1950-2016

	Destination of Cite:		
Origin:	Agri.	Manu.	Serv.
Agri.	0.57	0.38	0.05
Manu.	0.01	0.80	0.19
Serv.	0.00	0.69	0.30

- Agriculture draws from Manufacturing and Services.
  - Converse not true.

Evolution of US R&D Investment



Include Software

Structural Change in Sectoral TFP Growth

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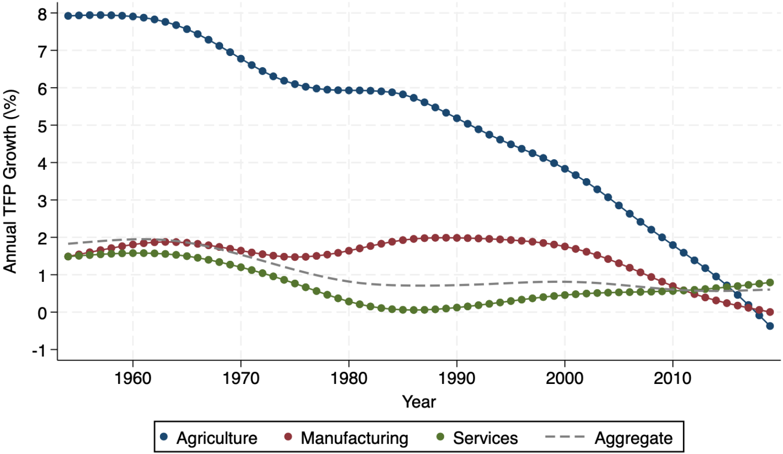

Similar for labor productivity.

Theory

Demographics, Preferences: Pop. Growth, CRRA+NHCES

- Homogeneous mass agents $L(t) \equiv L_o e^{\eta t}$ with $\eta \geq 0$.
- Agents' intertemporal utility:

$$\int_0^\infty e^{-(\rho-\eta)t} \frac{C(t)^{1-\vartheta} - 1}{1-\vartheta} dt, \qquad \text{with } \rho > \eta. \qquad (1)$$

$C(t)$ is consumption aggregator of sectoral goods $\{C_i(t)\}_{i=1}^I$.

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- Agents' intratemporal utility:

$$\sum_{i=1}^{I} \left(\frac{C_i(t)}{C^{\epsilon_i}(t)} \right)^{\frac{\sigma-1}{\sigma}} = 1, \tag{2}$$

  - $\epsilon_i > 0$, parametrizes income elasticity of sector i,
  - σ is the elasticity of substitution.
  - Assume complements $\sigma \in (0,1)$, and $\min_i\{\epsilon_i\} > 1 - \vartheta$.

Household Inter- and Intra-period Decisions Solution*

- Budget constraint (labor and capital income spent or saved):

$$\dot{\mathcal{A}} + E(t) \leq W(t) + r(t)\,\mathcal{A}(t) \tag{3}$$

HH Optimal Choices Given $[P(\cdot), r(\cdot), W(t)(\equiv 1)]_{t=0}^{\infty}, \mathcal{A}(0)$

Notation: $\bar{p} = \sum_{i=1}^{I} \Omega_i p_i$, $p_i = \ln P_i$ *except* for $r(t)$.

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Optimal paths $C(t), \{C_i(t)\}$ maximizing (**??**) s.t. (**??**) and (**??**) solve

$$\dot{c} \equiv \frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho-\dot{\bar{p}}(t)\left[1+(1-\sigma)Cov\left(\frac{\epsilon_i}{\bar{\epsilon}(t)},\frac{\dot{p}_i(t)}{\dot{\bar{p}}(t)};t\right)\right]}{\vartheta+\bar{\epsilon}(t)\left[1+(1-\sigma)Var\left(\frac{\epsilon_i}{\bar{\epsilon}(t)};t\right)\right]-1}, \tag{4}$$

$$\Omega_i(t) \equiv \frac{P_i(t)\,C_i(t)}{E(t)} = \left(\frac{P_i(t)}{E(t)}C(t)^{\epsilon_i}\right)^{1-\sigma} \quad \forall i, \tag{5}$$

$$E(t) = \left(\sum_{i=}^{I}\left(P_i(t)\,C(t)^{\epsilon_i}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, \tag{6}$$

plus transversality condition $\lim_{t\to\infty} e^{-(\rho-\eta)t}\frac{\mathcal{A}(t)}{E(t)}C(t)^{1-\vartheta}\frac{1}{\bar{\epsilon}(t)} = 0.$

Discussion of Properties the Household Behavior*

- Deviations Euler equation from homothetic case:
  - Term $\overline{\epsilon}\left(t\right)\left[1+\left(1-\sigma\right)Var\left(\frac{\epsilon_i}{\overline{\epsilon}(t)};t\right)\right]-1$ implies that the concavity of C (IES) depend on t: $p_i(t)$, $C(t)$.
  - Term $\left(1-\sigma\right)Cov\left(\frac{\epsilon_i}{\overline{\epsilon}(t)},\frac{\dot{p}_i(t)}{\dot{\overline{p}}(t)};t\right)$ consumption grows faster if prices fall faster for more income-elastic goods.
- Growth rates of c and e satisfy: [divisa index and line integral]

$$\overline{\epsilon_i}\left(t\right)\dot{c}\left(t\right)=\dot{e}\left(t\right)-\overline{\dot{p}_i}\left(t\right)$$

- Expenditure shares in sector i grow according to

$$\begin{aligned}\dot{\omega}_i\left(t\right)&=\left(1-\sigma\right)\left(\epsilon_i\dot{c}\left(t\right)+\dot{p}_i\left(t\right)-\dot{e}\left(t\right)\right),\\&=\left(1-\sigma\right)\left[\left(\epsilon_i-\overline{\epsilon}\left(t\right)\right)\dot{c}\left(t\right)+\dot{p}_i\left(t\right)-\overline{\dot{p}}\left(t\right)\right],\end{aligned}\qquad(7)$$

income and price effects at work.

Production is Multi-sector with Two Stages of Production

- Each sector i requires intermediate goods $v \in [0, 1]$.
- Final good producers in sector i produce

$$Y_i(t) = \left(\int_0^1 X_{iv}(t)^{\frac{\zeta}{\zeta+1}} \, dv \right)^{\frac{\zeta+1}{\zeta}}, \qquad \zeta > 0$$

  and $X_{iv}(t)$ is sector-specific intermediate input of type v.
- Each intermediate has a variety-specific productivity $Q_{iv}(t)$.
- Labor is only factor of production for intermediates

$$X_{iv}(t) = Q_{iv}(t) L_{iv}(t).$$

Innovation Occurs by Improving Intermediates*

- Integrate the BEJK(03) + Kortum(97) setup to demand side.
  - Delivers "standard" semi-endogenous growth model.
  - Delivers model mapping to patenting (incl. citations).

Innovation Occurs by Improving Intermediates*

- Integrate the BEJK(03) + Kortum(97) setup to demand side.
  - Delivers "standard" semi-endogenous growth model.
  - Delivers model mapping to patenting (incl. citations).
- Market for intermediates is as in BEJK (2003).
  - Varieties of each intermediate w/ Bertrand competition.
  - Innovation directed to a sector, improvement in random variety.
  - Keep track distribution productivities in each sector.
- Growth and patenting as in Kortum (1997).
  - Innovation done through costly R&D.
    - "Inspiration exists, but it has to find you working" (Picasso)
  - Patenting: byproduct of a successful innovation.
  - Citations: byproduct building from previous knowledge (in any sector).
- Details

Result: It Suffices to Study Innovation at Sectoral Level

- Only need to keep track of sectoral **stock of knowledge** K_i.

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- Only need to keep track of sectoral **stock of knowledge** K_i.
- Law of motion of $\{K_i\}_{i=1}^{I}$ is system coupled ODEs:

$$\dot{K}_i(t) = \Gamma_i Z_i(t)^{1-\alpha} S_i(t)^{\beta_i}, \tag{8}$$

where $\Gamma_i > 0$, $Z_i(t)$ are $R\&D$ researchers, $\alpha \in (0,1)$, $\beta_i > 0$.

- S_i denotes sectoral Knowledge spillovers, given by

$$S_i(t) \equiv \sum_{j=1}^{I} \Phi_{ij}^{-\theta} K_j(t), \tag{9}$$

$\Phi_{ij}^{-\theta} > 0$ is a knowledge applicability cost.

Patenting is a byproduct of improving a variety

- A patent is issued if firm takes over variety by margin $\Psi_i \geq 1$.
  - Ψ_i allows for patents being of different "quality" across sectors.
- Probability a patent in sector i cites (builds on) sector j:

$$Pr_{ij} = \frac{\Phi_{ij}^{-\theta} K_j(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(t)}. \tag{10}$$

  - Pr_{ij} increasing in stock of knowledge in j,
  - Pr_{ij} decreasing in applicability cost.
- Spillover S_i derived from probability an idea in i builds on j,

Sectoral Equilibrium Results (as in BEJK & BO)

- Equilibrium Definition
- The price index for the intermediate goods is

$$P_i(t) = \chi / K_i(t) \tag{11}$$

where χ is a constant independent of i.

- Aggregate sectoral profits constant share of revenue

$$\Pi_i(t) = \frac{1}{1+\theta} P_i(t) Y_i(t). \tag{12}$$

- Until being displaced by a future innovation, value of technique in i satisfies:

$$r(t) V_i(t) - \dot{V}_i(t) = \frac{\Pi_i(t)}{K_i(t)}. \tag{13}$$

Decomposition of the Relative Sectoral Profits

- Relative value of new ideas in two different sectors i and j

$$\frac{\Pi_i(t)}{\Pi_j(t)} = \frac{\Omega_i(t)}{\Omega_j(t)} = \overbrace{\left(\frac{K_j(t)}{K_i(t)}\right)^{1-\sigma}}^{\text{price effect}} \times \overbrace{C(t)^{(1-\sigma)\left(\epsilon_i - \epsilon_j\right)}}^{\text{income effect}} \tag{14}$$

- Two effects: relative price and income effects.
- The income effect term generates differential demand pull:
  - Income elasticities also shape innovation incentives.
  - If the demand for output of sector i is more income-elastic compared to sector j, the demand for the output of this sector grows relative to sector j as the households' aggregate consumption grows.

Free Entry and Market Clearing (to close the model)

- Arbitrage condition labor in R&D in sector i and production,

$$\Gamma_i Z_i(t)^{-\alpha} S_i(t)^{\beta_i} \; V_i(t) \leq 1 \equiv W(t),$$

note that since $\alpha \in (0,1)$ (and finite S_i), $Z_i(t) > 0$.

- Goods mkt clearing: total expenditure equals value of output,

$$L(t) E(t) = Y(t) = \sum_i P_i(t) Y_i(t),$$

- Labor markets clearing:

$$\sum_{i=1}^{I} \left(L_i^p(t) + Z_i(t) \right) = L^p(t) + Z(t) = L(t),$$

where $Z(t) = \sum_i Z_i(t)$.

Characterizing the Model Dynamics: Proceed in Two Steps

- Proceed in two steps:
  1. Characterize asymptotic behavior: constant growth path
  2. Simulate transitional dynamics in the calibrated model.

Characterizing the Model Dynamics: Proceed in Two Steps

- Proceed in two steps:
    1. Characterize asymptotic behavior: constant growth path
    2. Simulate transitional dynamics in the calibrated model.
- Intuition for model dynamics. Three moving parts:
    1. Consumption/saving decision: informed by Euler equation.
    2. Allocation of labor (sectoral $R\&D$ and production): reflects changes in value of innovation, informed by free entry + market clearing.
    3. Dynamics of sectoral knowledge: reflects state of the economy, informed by law of motion for knowledge.
- Detailed Dynamic Equations

Constant Growth Path (CGP): Definition

- A CGP is an allocation path in which aggregate consumption $C(t)$ and sectoral technologies asymptotically grow at constant rates,

$$\lim_{t\to\infty} \dot{c}(t) = g^* > 0, \tag{15}$$

$$\lim_{t\to\infty} \dot{k}_i(t) = \gamma_i g^*, \qquad \text{for } 1 \le i \le I, \tag{16}$$

where $(g^*, \gamma_1, \cdots, \gamma_I)$ are constant nonnegative values.

- We can define normalized asymptotic levels

$$C^* \equiv \lim_{t\to\infty} C(t)\, e^{-g^* t}, \tag{17}$$

$$K_i^* \equiv \lim_{t\to\infty} K_i(t)\, e^{-\gamma_i g^* t}. \tag{18}$$

CGP Characterization: Proceeds in Two Steps

- $\exists!$ candidate $\boldsymbol{\gamma} \equiv (\gamma_1, \cdots, \gamma_I) > 0$ and $g^* > 0$ satisfying equilibrium demand and supply CGP constraints (see below).
- Asymptotic growth knowledge spillovers $\gamma_i^S = \overline{\gamma} > 0$:

$$\overline{\gamma} = \frac{1}{1 + (1-\alpha)(1-\sigma)} \left(\max_j \{ (1-\alpha)(1-\sigma)\epsilon_j + \beta_j \overline{\gamma} \} + \max_i \{ \epsilon_i - \beta_i \overline{\gamma} \} \right).$$

- Sectors i^* with asymptotically nonnegligible shares of production and R&D employment satisfy $\epsilon_{i^*} - \beta_{i^*}\overline{\gamma} = \max_i \{ \epsilon_i - \beta_i \overline{\gamma} \}$.
- If $\epsilon_{i^*} - \beta_{i^*}\overline{\gamma} > (\vartheta - 1)(1-\alpha)\eta / (\rho - \eta)$, $\exists!$ CGP with

$$g^* = \frac{(1-\alpha)\eta}{\epsilon_{i^*} - \beta_{i^*}\overline{\gamma}} > 0, \qquad r^* = \rho + (\vartheta - 1) g^*,$$

$$\gamma_i = \frac{1}{1 + (1-\alpha)(1-\sigma)} \left[(\beta_i - \beta_{i^*}) \overline{\gamma} + (1-\alpha)(1-\sigma)\epsilon_i + \epsilon_{i^*} \right].$$

- For all $i \in \mathcal{I}^*$, $\gamma_i = \epsilon_i$

Sectoral Growth along the CGP: Demand Pull, Tech. Push

- Differences in productivity growth between sectors i and j:

$$\gamma_i - \gamma_j \propto \underbrace{(\beta_i - \beta_j)\,\overline{\gamma}}_{\text{Tech. Push}} + \underbrace{(1-\alpha)\,(1-\sigma)\,(\epsilon_i - \epsilon_j)}_{\text{Demand Pull}}.$$

- Cross-sector variation in the rates of productivity growth:
  1. Technology Push: mediated by spillovers $\{\beta_i\}$.
  2. Demand Pull: mediated by nonhomotheticity params. $\{\epsilon_i\}$.

- More on Demand and Supply Constraints

Quantification and Model Dynamics

Proceed in Three Steps to Quantify Model

- To pin down model parameters:
  1. Use "off-the-shelf" for parameters already estimated in the literature (prefs. elast, discount factor, IES, markups)
  2. Use model structure to estimate innovation parameters.
  3. Calibrate initial conditions of Knowledge Stock $(K_i)_{i \in \mathcal{I}}$ and constant taste and innov. productivity through matching evolution of
     - Sectoral VA shares.
     - Sectoral R&D shares.
     - Sectoral TFP growth.
- After quantifying our model
  - Quantify model CGP: Closed form Expressions.
  - Assess model dynamics simulating system ODEs and do counterfactuals.

Other Params. needed for CGP: Calibrated Outside Model

- Income elasticity parameters and ES across sectors (CLM 21):

$$\epsilon_a = 0.06, \ \epsilon_m = 1, \ \epsilon_s = 1.6, \ \sigma = 0.5.$$

- Other parameters "standard":

$$\begin{aligned} \text{CRRA} : & \ \vartheta = 1.5 \\ \text{Discount} : & \ \rho = 0.02 \\ \text{Markups} : & \ \zeta = 3 \ (\longrightarrow \text{Markup} = 4/3) \\ \text{Frechet/Pareto Tail} : & \ \theta = 1.45 \\ \text{Population growth} : & \ \eta = 0.015 \\ \text{Innovation Congestion} : & \ \alpha = 0.6 \end{aligned}$$

We Estimate Innovation Parameters with US Patent Data

- Use model equations to estimate $\{\psi_j, \beta_j, \phi_{ij}\}_j$
- From theory we know that $\dot{K}_j/K_j = \psi_j \cdot \text{patents}_j$

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- Use model equations to estimate $\{\psi_j, \beta_j, \phi_{ij}\}_j$
- From theory we know that $\dot{K}_j / K_j = \psi_j \cdot \text{patents}_j$
- Probability of citations between origin j and destination i

$$Pr_{ij}(t) = \frac{\Phi_{ij}^{-\theta} K_j(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(t)}.$$

- Gravity: logs + origin-destination and destination-time FEs:

$$\Delta \ln Pr_{ij}(t) = \psi_j \ln \text{patents}_j(t) + \delta_{it} + \epsilon_{ijt}$$

- Estimate ψ_j by two subsamples:
  - Number of patents has grown substantially since 1980s
  - Estimates post 1980 one to two orders of magnitude smaller
- We back out $\Delta \ln K_i$ with patents flow and ψ_j

Model Parametrization: Obtaining elasticity of spillovers

- Model implies that:

$$\beta_i = \frac{\Delta \ln \left(\frac{\dot{K}_{it}}{K_{it}}\right) + \Delta \ln K_{it} - (1-\alpha)\Delta \ln Z_{it}}{\Delta \ln S_i}$$

  recall we calibrate $\alpha = 0.6$ and

$$\Delta \ln \left(\frac{\dot{K}_{it}}{K_{it}}\right) = \Delta \ln \left(\psi^{-\theta} pat_i\right) = \ln \left(\frac{\psi_{post}^{-\theta}}{\psi_{pre}^{-\theta}}\right) + \ln \left(\frac{pat_{i,post}}{pat_{i,pre}}\right)$$

$$\Delta \ln K_{it} = \psi_{pre}^{-\theta} pat_{i,pre} + \psi_{post}^{-\theta} pat_{i,post}$$

$$\Delta \ln S_i = \psi_{pre}^{-\theta} \sum_j Citations_{ij,pre} + \psi_{post}^{-\theta} \sum_j Citations_{ij,post}$$

- So we can compute these by hand, to obtain:

$$\beta_a = 0.87$$
$$\beta_m = 0.69$$
$$\beta_s = 0.45$$

- Normalize $\phi_{ii} = 1$, estimate ϕ_{ij}, $i \neq j$ using Pr_{ij} & $\Delta \ln K_i$.

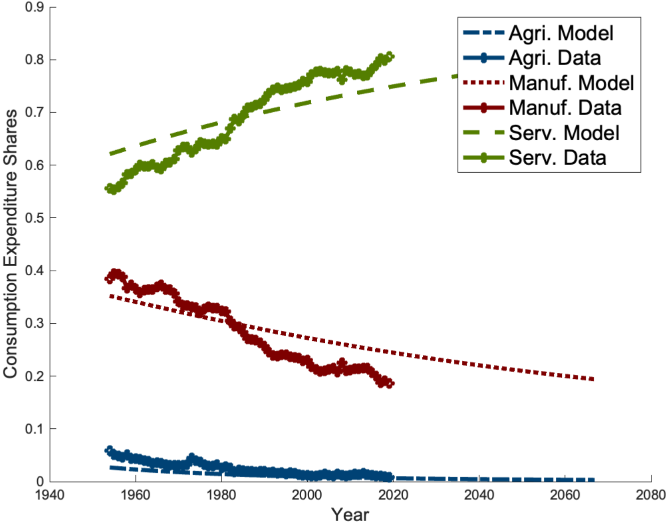
Third Step of Calibration: Match along Transition

- Take the path from 1950 to 2010 of
  - Value added shares
  - R&D Shares
  - Sectoral TFP growth
- Guess and iterate over nine parameters:
  - Initial conditions for knowledge stock in 1950, (K_i)
  - Constant productivity term in innovation, (Γ_i)
  - Constant taste shifters across sectors in preferences.
- Backward shooting from perturbed CGP along smallest eigenvec.
  - By construction, ensures convergence going forward.

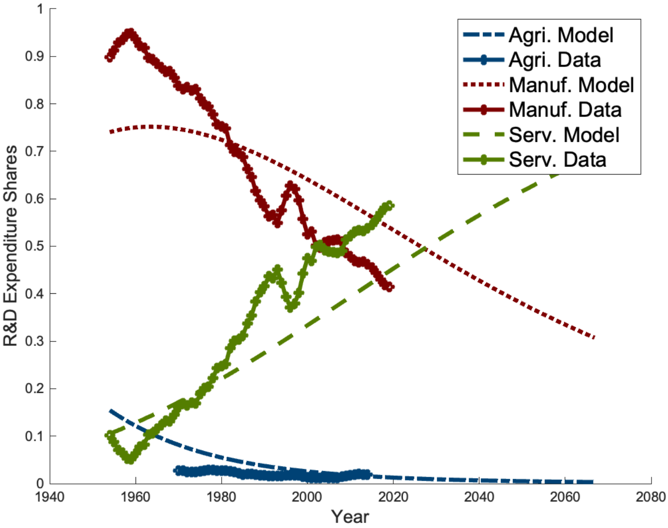
Model Dynamics: Looking backward and forward

- Model delivers endogenously joint evolution of:
  1. Sectoral TFP growth.
  2. Structural Change in production/employment
  3. Structural Change in R& D.
- Model generates very rich dynamics, a priori many different things can happen.
- Three exercises:
  1. Evaluate performance in accounting for past (model fit)
  2. Evaluate future productivity: slowdown, Baumol's disease cost.
  3. Counterfactuals

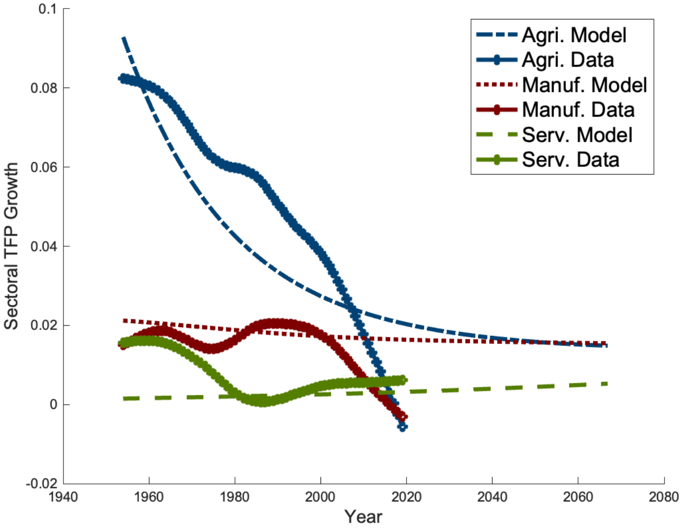
Calibrated model: Sectoral Shares



Calibrated model: R& D Shares

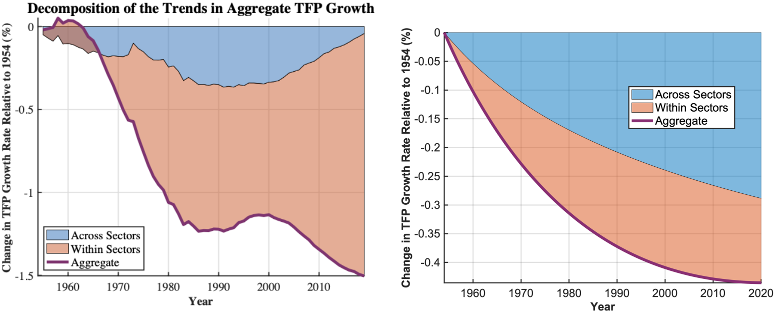


Calibrated model: TFP Growth



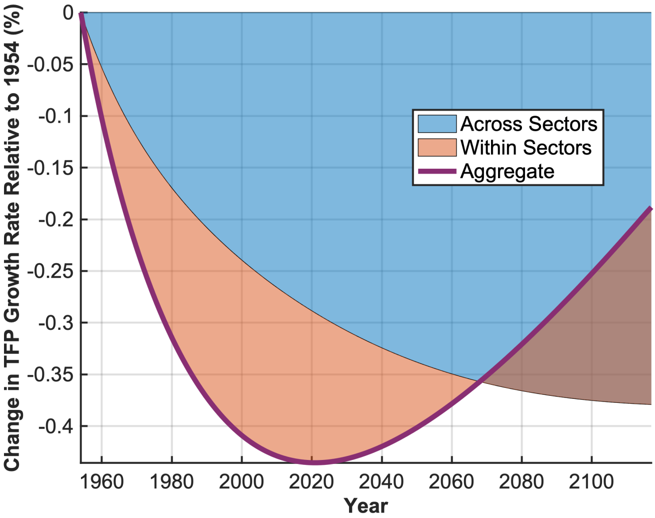
Within-Across Decomposition: Data vs Model

$$\Delta g_t = \sum_i \Delta S_{it}\overline{g}_{it} + \sum_i \overline{S}_{it}\Delta g_{it},$$

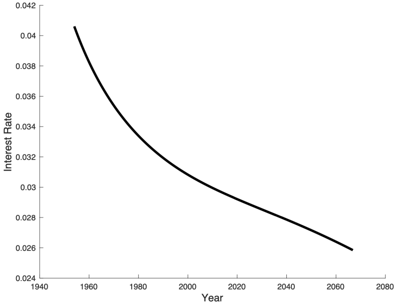


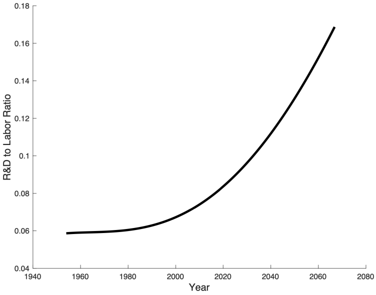
Decomposing (future) Productivity Slowdown (?)

$$\Delta g_t = \sum_i \Delta S_{it} \overline{g}_{it} + \sum_i \overline{S}_{it} \Delta g_{it},$$



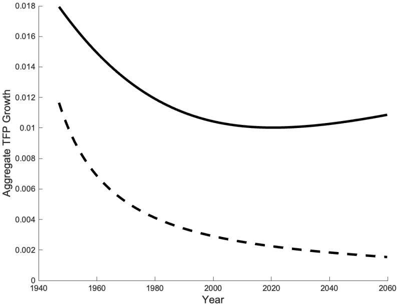
Aggregates: Interest Rate and Share of workers in R&D



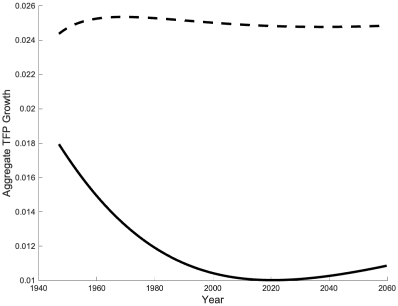


Understanding the Mechanism: Effects on Aggregate TFP of

- $\downarrow$ Nonhomotheticity in S

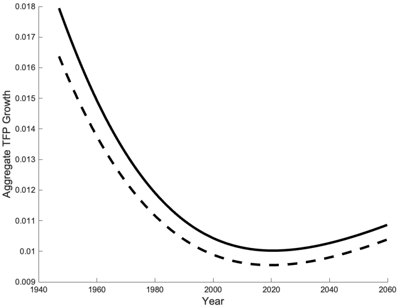


- $\uparrow \beta_s$ to be as β_m

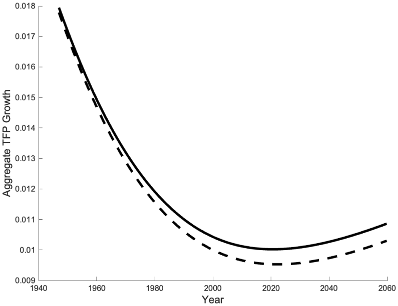


Effect of a negative 10% productivity shock to R&D

- Uniform to all Sectors



- To only Services



Conclusions

- Documented Structural Change in Innovation.
- Presented theory of sectoral directed technical change.
- Long-run growth rate innovation modulated by income effect and sectoral differences in innovation technology.
- Quantification of the model:
  - Constant Growth Path: Long run growth of 0.9%
  - Showed (preliminary) calibration of model dynamics, account for substantial part of variation in the data.
  - Calibration generates comov't Output Shares and TFP growth.
- End goals: Analyze prod. slow down and rate struct. change.

Calibrated Constant Growth Path

- Estimated $\{\beta_i, \alpha\} + \{\epsilon_i\} \Longrightarrow$ Services dominate in long-run.
- Values along the CGP:

$$g^* = 0.96\%,$$
$$r^* = 2.48\%,$$
$$\frac{Z}{L} = 26\%. \quad \text{Employment Share in R\&D (!)}$$

- DHV (22) simulate nested NHCES w/ exogenous TFP
  - $\simeq$ baseline scenario of 0.9% growth in distant future (2079-2089)
  - Remarkable since we use completely different data
- Calibrated differential growth, tech. push and demand pull,

$$\gamma_i - \gamma_j \propto \underbrace{(\beta_i - \beta_j)\,\overline{\gamma}}_{\text{Tech. Push}} + \underbrace{(1-\alpha)\,(1-\sigma)\,(\epsilon_i - \epsilon_j)}_{\text{Demand Pull}}.$$

  - $(\gamma_a - \gamma_s)g^* = 0.41\%$
  - $(\gamma_m - \gamma_s)g^* = 0.55\%$

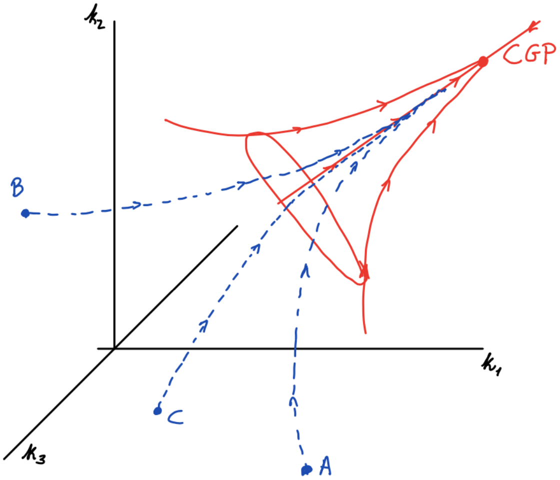
Dynamics around CGP

- Dynamics $\neq$ standard 1-sector neoclassical growth model

$$\begin{pmatrix} \dot{c} \\ \dot{k}_a \\ \dot{k}_m \\ \dot{k}_s \\ \dot{z}_a \\ \dot{z}_m \\ \dot{z}_s \end{pmatrix} = \mathbf{J} \begin{pmatrix} c \\ k_a \\ k_m \\ k_s \\ z_a \\ z_m \\ z_s \end{pmatrix} + \begin{pmatrix} c^* \\ k_a^* \\ k_m^* \\ k_s^* \\ z_a^* \\ z_m^* \\ z_s^* \end{pmatrix}$$

- 3D Stable Manifold ($\mathbf{J}$ has 3 negative roots, 4 positive).
  - Three stock variables in the model.
- Slower conv. than neoclass. grwth model: 1/2 life 100 yrs
  - Consistent with Buera et al. (2021) "STraP" paper.
  - Half-life: $\frac{\ln 2}{\gamma_a - \gamma_s} \sim 70$ yrs. $\gg$ half-life NGM $\sim 5 - 7$ yrs.

3-Dimensional Stable Manifold



Thank You!

Questions or comments?

marti.mestieri@gmail.com

Appendix Slides

Extensions

1. Alternative model of spillovers.

Alternative Formulation of Spillovers

- Do these results depend on unbounded support for ideas?
  - No.
  - Alternative model: similar CGP properties and dynamics.
- Expanding variety model with

$$Y_i(t) = \left(\int_0^{N_i} X_{iv}(t)^{\frac{\zeta}{\zeta+1}} dv\right)^{\frac{\zeta+1}{\zeta}}.$$

- Or Schumpeterian formulation with N_i being average quality.

Assumptions on Innovation Technology

$$\dot{N}_i(t) = \frac{1}{\eta_i} S_i(\mathbf{N}(t)) Z_i(t),$$

- $\partial S_i / \partial N_j \geq 0$ for all i and j.
- Each S_i is homogenous of degree 1 in its arguments
- The following limit exists and satisfies

$$\lim_{N_i \to \infty} \frac{S_i(\mathbf{N})}{N_i} > 0,$$

- The matrix $[\Sigma_{ij}] \equiv \left[\frac{\partial \log S_i}{\partial \log N_j}\right]_{ij}$ is positive definite.

Example of S_i

- Nested CES

$$S_i(\mathbf{N}) \equiv \frac{1}{\eta_i}\left[\delta_i^{1-\psi_i} N_i^{\psi_i} + (1-\delta_i)^{1-\psi_i}\,\tilde{S}_i(\mathbf{N})^{\psi_i}\right]^{\frac{1}{\psi_i}},$$

$$\tilde{S}_i(\mathbf{N}) \equiv \left(\sum_{j\neq i} \vartheta_{ij}^{1-\varsigma_i} N_j^{\varsigma_i}\right)^{\frac{1}{\varsigma_i}},$$

$\tilde{S}(t)$: an economy-wide, general purpose stock of knowledge

More

Replacement of (New) Techniques*

- Probability productivity new technique exceeds frontier in i

$$\int_0^\infty \int_0^\infty \left(\frac{Q}{\underline{Q} Q_o^{\beta_i}} \right)^{-\theta} d\widetilde{F}_i (Q_o, t) \; dF_i (Q, t) = \underline{Q}^\theta \Gamma (1 - \beta_i) \frac{S_i (t)^{\beta_i}}{K_i (t)}. \tag{19}$$

  - Prev. step: prob. productivity new technique exceeds Q

$$\int_0^\infty \left(\frac{Q}{\underline{Q} Q_o^{\beta_i}} \right)^{-\theta} d\widetilde{F}_i (Q_o, t) = \underline{Q}^\theta \Gamma (1 - \beta_i) \times S_i (t)^{\beta_i} \; Q^{-\theta}.$$

- Since rate of arrival of new ideas is $\widetilde{\Gamma}_i Z_i (t)^{1-\alpha}$, rate of displacement of frontier techniques in sector i, $\dot{K}_i (t) / K_i (t)$.
- After a new technique improves on the frontier of a variety, it can get replaced at rate $\dot{K}_i (\tau) / K_i (\tau)$ for all $\tau \geq t$
- Probability this new technique survives til $t' > t$ is:

$$\exp \left(- \int_t^{t'} \frac{\dot{K}_i (\tau)}{K_i (\tau)} d\tau \right) = \frac{K_i (t)}{K_i (t')}. \tag{20}$$

Back.

CGP - Demand Side Constraints

- Along CGP,

$$\lim_{t\to\infty} \Omega_i(t) = \Omega_i^*,$$
$$\lim_{t\to\infty} e^{-\eta t} L^p(t) = L^* > 0.$$

- Let $\mathcal{I}^*$ denote the set of industries such that $\Omega_i^* > 0$, and assume $g^* > 0$.
- Result 1: for any industry $i \in \mathcal{I}^*$, the asymptotic rate of productivity growth in the sector satisfies $\gamma_i = \epsilon_i, \qquad i \in \mathcal{I}^*$.
- Result 2: For any industry $i \notin \mathcal{I}^*$, we have the condition $\gamma_i > \epsilon_i$ if $\sigma \in (0,1)$ and $\gamma_i < \epsilon_i$ if $\sigma \in (1,\infty)$.

CGP - Demand Side Constraints

- By CGP definition, production shares fall at the rate

$$\lim_{t\to\infty} \dot{\omega}_i(t) = -(1-\sigma)(\gamma_i - \epsilon_i) g^* \leq 0,$$

with the inequality being strict for $i \notin \mathcal{I}^*$.

- Asymptotic shares:

$$\Omega_i^* \equiv \lim_{t\to\infty} \Omega_i(t) e^{\xi_i g^* t} = \left(\frac{\chi (C^*)^{\epsilon_i}}{E^* K_i^*} \right)^{1-\sigma},$$

- Total consumption expenditure asymptotically given by

$$E^* = \left(1 + \frac{1}{\theta}\right) \frac{L^*}{L_o} = \left[\sum_{i\in\mathcal{I}^*} \left(\frac{\chi (C^*)^{\epsilon_i}}{K_i^*} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \in \left(0, 1 + \frac{1}{\theta}\right),$$

CGP - Supply Side Constraints

- Define the asymptotic rate of growth innovation spillovers to i

$$\lim_{t\to\infty} \dot{s}_i(t) = \gamma_i^S g^*.$$

with $\gamma_i^S = \gamma_{j^*(i)},\ j^*(i) \equiv \underset{j}{\text{argmax}} \left\{ \frac{\eta/g^* + (1-\sigma)\epsilon_j}{1-\beta_j+(1-\alpha)(1-\sigma)} \right\}$

- Asymptotic rate of productivity growth in i:

$$\gamma_i = \frac{1}{1+(1-\alpha)(1-\sigma)} \left[\beta_i \gamma_i^S + (1-\alpha)\frac{\eta}{g^*} + (1-\alpha)(1-\sigma)\epsilon_i \right].$$

- For any sector i with $\lim_{t\to\infty} Z_i(t) > 0$,

$$\lim_{t\to\infty} \dot{z}_i(t) = \eta - (1-\sigma)(\gamma_i - \epsilon_i) g^* \leq \eta.$$

- Let $\mathcal{I}^\dagger$ denote the set of sectors that asymptotically constitute a nonnegligible share of R&D expenditures.
- For $i \in \mathcal{I}^\dagger$, the expression is satisfied with equality and we have $\gamma_i \equiv \epsilon_i$.

CGP – Supply Side Constraints: Implications

- Asymptotic rate of technological growth rises linearly in the rate of spillovers and in the income elasticity of sectors.

→ Technological growth is asymptotically faster in sectors with higher income elasticities ϵ_i.

- Intuition from free entry + nonhomotheticity:

$$1 = \underbrace{S_i(t)^{\beta_i}}_{\text{Spillovers}} \times \underbrace{Z_i(t)^{-\alpha}}_{\text{DRS to R\&D Inputs}} \times \overbrace{\underbrace{\frac{1}{K_i(t)}}_{\text{Competition Effect}} \times \underbrace{\frac{L_o e^{\eta t} \cdot E^*}{r_i^*} \Omega_i(t)}_{\text{Market Size Effect}}}^{\text{Value of Innovation}},$$

- Sectors that asymptotically constitute a nonzero share of production and R&D indeed coincide, i.e., $\mathcal{I}^* = \mathcal{I}^\dagger$.

Back

R&D Market Free Entry Condition

- $V_i(t)$: value of owning intermediate input firm in sector i:

$$R(t)V_i(t) - \dot{V}_i(t) = \Pi_i(t)$$

- Free entry condition:

$$\text{wage} = 1 = \underbrace{\frac{S_i(t)}{\eta_i}}_{\text{innovative productivity of labor}} \times V_i(t)$$

- Rewrite as:

$$\underbrace{N_i(t)V_i(t)}_{\text{total assets in } i} = \underbrace{\eta_i \frac{N_i(t)}{S_i(t)}}_{\text{cost of growth } i} = \frac{Z_i(t)}{\dot{N}_i(t)/N_i(t)}$$

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- Define *Sectoral Share of Total Corporate Assets:*

$$\Lambda_i(t) \equiv \frac{N_i(t)\,V_i(t)}{\sum_j N_j(t)\,V_j(t)}$$

Characterization for CGP with $0 < \sigma < 1$

Sectoral Innovation Growth

- The asymptotic growth rates in $\mathcal{I}^*$ is

$$\gamma_i = \xi \epsilon_i.$$

- Sectoral Innovation Growth near CGP with example spillovers

$$\gamma_i = \xi \epsilon_i \left(1 + \frac{\xi_i}{\xi_i + 1 - \sigma} \left(\frac{\epsilon_{max}}{\epsilon_i} - 1 \right) \right),$$

where $\xi_i \equiv \xi$ if $\psi_i > 0$ and $\xi_i \equiv \xi (1 - \delta_i)$ if $\psi_i \to 0$. Details

  - Vanishing sector has higher productivity growth (services vs. manufacturing).

Results for the Particular Specification of Spillovers*

- Recall Nested CES Structure

$$
\begin{aligned}
S_i(\mathbf{N}) &\equiv \frac{1}{\eta_i}\left[\delta_i^{1-\psi_i} N_i^{\psi_i} + (1-\delta_i)^{1-\psi_i}\,\tilde{S}_i(\mathbf{N})^{\psi_i}\right]^{\frac{1}{\psi_i}}, \\
\tilde{S}_i(\mathbf{N}) &\equiv \left(\sum_{j\neq i} \vartheta_{ij}^{1-\varsigma_i} N_j^{\varsigma_i}\right)^{\frac{1}{\varsigma_i}},
\end{aligned}
$$

Non-vanishing set of Sectors $\mathcal{I}^*$

Set of sectors that asymptotically constitute a nonvanishing share of economic activity $\mathcal{I}^*$ consists of

1. Any sector i with $\varsigma_i > 0$ and $\psi_i < 0$, or $\varsigma_i < 0$ and $\psi_i > 0$
2. Any sector i with $\varsigma_i < 0$ and $\psi_i < 0$ if $\epsilon_i \leq \epsilon_{i'}$ for all i' ,
3. Any sector i with $\varsigma_i > 0$ and $\psi_i \geq 0$ if $\epsilon_i \geq \epsilon_{i'}$ for all i'.

Results for the Particular Specification of Spillovers* (ct'd)

- First Order Approx. Growth near CGP:

$$\gamma_i = \xi \epsilon_i \left(1 + \frac{\xi_i}{\xi_i + 1 - \sigma} \left(\frac{\epsilon_{max}}{\epsilon_i} - 1 \right) \right), \qquad (21)$$

  where $\xi_i \equiv \xi$ if $\psi_i > 0$ and $\xi_i \equiv \xi (1 - \delta_i)$ if $\psi_i \to 0$.

- Vanishing sector has higher productivity growth.
  - Manufacturing vs. Services

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- First Order Approx. Growth near CGP:

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  where $\xi_i \equiv \xi$ if $\psi_i > 0$ and $\xi_i \equiv \xi (1 - \delta_i)$ if $\psi_i \to 0$.
- Vanishing sector has higher productivity growth.
  - Manufacturing vs. Services
- Furthermore the total value of assets in sector $i \in \mathcal{I}^*$ asymptotically converges to

$$\Lambda_i^* = \eta_i \delta_i^{\frac{\psi_i - 1}{\psi_i}}. \qquad (22)$$

Results for the Particular Specification of Spillovers* (ct'd)

Equilibrium Characterization

Let $A^* \equiv \frac{1}{H}\sum_{i\in\mathcal{I}^*}\Lambda_i^*$ and denote by $\langle\epsilon_i\rangle^*$ and $Var\,\langle\epsilon_i\rangle^*$ under $\{\Lambda_i\}$. Suppose $\frac{1}{\zeta A^*}\left(1-\frac{\theta+\langle\epsilon_i\rangle^*-1+(\zeta+1)\langle\epsilon_i\rangle^*}{1+(\zeta+1)\langle\epsilon_i\rangle^*}\right)<\rho<\frac{1}{\zeta A^*}$. Then CGP exists and is unique. Determined by

$$
\begin{aligned}
\bar{\epsilon}^* &= \langle\epsilon_i\rangle^* + \frac{Var\,\langle\epsilon_i\rangle^*}{r^*+\zeta\,\langle\epsilon_i\rangle^*},\\
r^* &= \frac{1}{\zeta A^*} - \frac{(1+\zeta)\,\langle\epsilon_i\rangle^*}{\bar{\epsilon}^*+\theta-1+(1+\zeta)\,\langle\epsilon_i\rangle^*}\left(\frac{1}{\zeta A^*}-\rho\right).\\
g^* &= \frac{1/\zeta A^*-\rho}{\theta+\bar{\epsilon}^*-1+(\zeta+1)\,\langle\epsilon_i\rangle^*},\\
\frac{L^*}{H} &= \frac{\rho\zeta A^*\,\langle\epsilon_i\rangle^*+\theta+\bar{\epsilon}^*-1+\zeta\,\langle\epsilon_i\rangle^*}{\theta+\bar{\epsilon}^*-1+(\zeta+1)\,\langle\epsilon_i\rangle^*}.\\
\Omega_i^* &= \frac{\zeta}{L^*}\left(r^*+\zeta g^*\epsilon_i\right)\Lambda_i^*, \qquad \text{for all } i\in\mathcal{I}^*.
\end{aligned}
$$

Back

Market of Intermediates (is as in BEJK 2003)

- $\forall t$: countable # of techniques to produce intermediate iv.
- Each technique owned by a producer, with productivity Q.
- Producers in each variety engage in Bertrand competition:
  - Price depends on gap b/w highest and second highest Q,
  - Either monopoly or limit pricing for any given iv and t.
- Frontier across varieties cumulative distribution: $F_i(Q, t)$.
  - Also need to keep track of distrib. of second highest Q.

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- Frontier across varieties cumulative distribution: $F_i(Q,t)$.
  - Also need to keep track of distrib. of second highest Q.
- Innovation will make frontier $F_i(Q,t)$ evolve.
  - Patenting: by-product of a successful innovation.
- Spoiler alert: BEJK$\Rightarrow 1/P_i$ related to mean of $F_i(Q,t)$.

Innovation and Patenting: R&D Technology

- Innovation done through costly R&D.
  - "Inspiration exists, but it has to find you working" (Picasso)

Innovation and Patenting: R&D Technology

- Innovation done through costly R&D.
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- Atomistic firms: firm $f \in \mathcal{F}_i(t)$ hires $Z_{if}(t)$ R&D workers.
  - Generates new ideas at a Poisson flow rate

$$\widetilde{\Gamma}_i \, Z_i\left(t\right)^{-\alpha} \, Z_{if}\left(t\right), \qquad \text{with} \quad \alpha \in (0,1).$$

  - Congestion: $Z_i(t) \equiv \int_{f \in \mathcal{F}_i} Z_{if}(t)\, df$ total mass R&D workers.
  - Note: Congestion term formulation will ensure $Z_i > 0$.
- Each new idea leads to new technique applicable to random v
  - where v is drawn from $U[0,1]$.
- From Bertrand assumption, if technique is the most productive, firm produces until being replaced by better one.
- Otherwise, it does not produce.

Innovation: Generation of Production Techniques

- Productivity of new technique Q' combines new and old Q's

$$Q' = Q_{new} \times Q_{old}^{\beta_i}, \qquad \text{with} \quad \beta_i > 0.$$

1. Q_{new} drawn from *exogenous* Pareto dist., tail param θ.
2. Q_{old} *adopted* from existing technique (β_i: strength spillovers).

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  1. Q_{new} drawn from *exogenous* Pareto dist., tail param θ.
  2. Q_{old} *adopted* from existing technique (β_i: strength spillovers).

- Find Q_{old} by drawing *one* frontier technique $\tilde{Q}_{old,j}$ at random (uniform distrib) from one of the varieties in each sector j.
- Then choose the best technique Q_{old} to build upon

$$Q_{old} = \max_{j=1,\cdots,I} \left\{ \frac{\widetilde{Q}_{oj}}{\Phi_{ij}} \right\}, \tag{23}$$

$\Phi_{ij} \geq 1$ cost applying idea from sector j to i ($\Phi_{ii} \equiv 1 \ \forall i$).

Innovation: Dynamics of Techniques Distribution(BO 21)

- Let $\widetilde{F}_i\left(Q_{old}, t\right)$ be cumul. distrib. of adopted techniques Q_{old}.
- LOM of frontier techniques in sector i:

$$\frac{\partial \log F_i\left(Q, t\right)}{\partial t} = -\widetilde{\Gamma}_i \underline{Q}^{\theta} Z_i\left(t\right)^{1-\alpha} Q^{-\theta} \int_0^{\infty} dQ_o Q_o^{\beta_i \theta} \widetilde{f}_i\left(Q_o, t\right) \tag{24}$$

  - Probability that no new technique in i exceeds Q is

$$F_i(Q, t + dt) = F_i(Q, t) e^{-\tilde{\Gamma}_i Z_i^{1-\alpha}(1-\hat{F}(Q,t))dt}$$

  - Combine with Pareto assumption $\mathbb{P}[Q_n > Q] = (Q/\underline{Q})^{-\theta}$.

- Define $K_i\left(t\right)$ as a measure of the stock of knowledge.

$$K_i\left(t\right) \equiv \widetilde{\Gamma}_i \underline{Q}^{\theta} \int_{-\infty}^{t} d\tau \; Z_i\left(\tau\right)^{1-\alpha} \int_0^{\infty} dQ_o \; Q_o^{\beta\theta} \widetilde{f}_i\left(Q_o, \tau\right). \tag{25}$$

  - Summary of the accumulated "push to the right of F_i."

Innovation: Dynamics of Production Techniques II

Assumption on Initial Distributions

The initial distribution of frontier techniques Fréchet distribution,

$$F_i\left(Q,0\right) \equiv \exp\left(-K_i\left(0\right) Q^{-\theta}\right) \qquad \text{for all} \quad i. \tag{26}$$

- Exploit max-stability (θ is *not* indexed by i) in (**??**) to obtain

$$\widetilde{F}_i\left(Q_o,t\right) = \exp\left[-S_i\left(t\right)\ Q_o^{-\theta}\right],$$

  where stock of knowledge spillovers is

$$S_i\left(t\right) \equiv \sum_{j=1}^{I} \Phi_{ij}^{-\theta} K_j\left(t\right).$$

- LOM in of Production Techniques (**??**) remains Fréchet $\forall t$:

$$F_i\left(Q,t\right) \equiv \exp\left(-K_i\left(t\right) Q^{-\theta}\right). \tag{27}$$

Innovation: Dynamics of Production Techniques III

- LOM of Production Techniques (**??**) becomes ODE in $\{K_i\}_{i=1}^{I}$:

$$\dot{K}_i(t) = \Gamma_i \, Z_i(t)^{1-\alpha} \; S_i(t)^{\beta_i}, \qquad (28)$$

  where $\Gamma_i \equiv \widetilde{\Gamma}_i \underline{Q}^{\theta} \Gamma(1-\beta_i)$.
- We observe $R\&D$ researchers $Z_i(t)$: useful for quantification.
- Bonus: Fréchet yields tractability of joint distribution 2 largest draws.
  - Closed form expression for sectoral price indices.

- More on the replacement of new techniques Go

Patenting is a byproduct of improving a Technique

- Patent issued if firm takes over variety by margin $\Psi_i \geq 1$.
- Rate of arrival new patents in i:
  arrival rate new techniques × quality above threshold

$$\Gamma_i Z_i(t)^{1-\alpha} \int_0^\infty \int_0^\infty \left(\frac{Q\,\Psi_i}{Q_o^{\beta_i}} \right)^{-\theta} d\widetilde{F}_i(Q_o, t)\, dF_i(Q, t) = \Psi_i^{-\theta} \times \frac{\dot{K}_i(t)}{K_i(t)}.$$

  where $\frac{\dot{K}_i(t)}{K_i(t)}$ rate of displacement of frontier techniques. Go

- Probability a patent cites (builds on) sector j:

$$\mathbb{P}\left(Q_o \equiv \widetilde{Q}_{oj}/\Phi_{ij} \text{ in sector } i\,, t \right) = \frac{\Phi_{ij}^{-\theta} K_j(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(t)}. \qquad (29)$$

  - Probability increasing in stock of knowledge in j,
  - decreasing in applicability cost.
- Back

Definition of the Market Equilibrium Back

- Define as allocation the collection of the time paths of aggregate and sector consumptions of households $[C(t), \boldsymbol{C}(t)]_{t=0}^{\infty}$, employment in production and R&D in each sector $[\boldsymbol{L}^p(t), \boldsymbol{Z}(t)]_{t=0}^{\infty}$, the stocks of knowledge of intermediate varieties in each sector $[\boldsymbol{K}(t)]_{t=0}^{\infty}$, and the expected value functions $[\boldsymbol{V}(t)]_{t=0}^{\infty}$ for each sector.
- An equilibrium is given by price vector $[\boldsymbol{P}(\cdot), r(\cdot), W(t)]_{t=0}^{\infty}$ and an allocation satisfying the constraints imposed by household utility maximization, monopolist profit maximization of intermediate producers, cost minimization of sectoral producers, and the free entry condition in innovation everywhere along the time paths. Note that these constraints take as given the vector of prices.
- Moreover, the price vector is such that, in equilibrium, the sectoral and aggregate consumption of households equals the sectoral supplies, and assets satisfy $A_i \cdot L = \sum_i K_i V_i$. Employment allocations are such that sectoral labor demand and R&D demand equals supply and total labor demand equals total labor supply.

Characterizing the Model Dynamics

- Initial vector stock of ideas $\boldsymbol{K}(0) = (K_1(0), \cdots, K_I(0))$.
- Evolution economy is:

$$\dot{c}(t) \equiv \frac{\dot{C}(t)}{C} = \mathcal{F}(C(t), \boldsymbol{K}(t)), \qquad \text{[Euler Eq.]} \qquad (30)$$

$$\dot{k}_i(t) \equiv \frac{\dot{K}_i(t)}{K_i(t)} = \mathcal{G}_i(C(t), \boldsymbol{K}(t)), \qquad \text{for } 1 \leq i \leq I, \quad (31)$$

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- To characterize $\mathcal{G}_i$, introduce variables*:
  1. Demand Pull (total value of assets i): $A_i(t) \equiv K_i(t) V_i(t)$.
  2. Technology Push: $B_i(t) \equiv \Gamma_i \frac{S_i(t)^{\beta_i}}{K_i(t)}$.
- Dynamics of knowledge stocks separable b/w input and output

$$Z_i(t) = A_i(t)^{\frac{1}{\alpha}} B_i(t)^{\frac{1}{\alpha}}, \tag{32}$$

$$\dot{k}_i(t) = A_i(t)^{\frac{1-\alpha}{\alpha}} B_i(t)^{\frac{1}{\alpha}}. \tag{33}$$

Dynamics of Knowledge Stocks*

- Dynamics technology push:

$$\dot{b}_i(t) \equiv \frac{\dot{B}_i(t)}{B_i(t)} = \beta_i \sum_j \Sigma_{ij}(t)\ \dot{k}_j(t) - \dot{k}_i(t), \tag{34}$$

where

$$\Sigma_{ij}(t) \equiv \frac{A_j \partial S_i}{S_i \partial A_j} = \frac{\Phi_{ij}^{-\theta} K_j(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(t)} \tag{35}$$

- Elasticity of spillovers in sector i with respect of the stock of knowledge in sector j is given by the share of sector-i patents citing patents from sector j at time t.

Dynamics of Knowledge Stocks II*

- Dynamics demand pull:

$$r(t)\, A_i(t) - \dot{A}_i(t) = \frac{L(t)}{1+\theta} E(t)\, \Omega_i(t) - Z_i(t)\,, \qquad (36)$$

- Connection between the pull force $A_i(t)$ and the demand side: the push force rises for sector i to the extent that the current share of household expenditure $\Omega_i(t)$ is large relative to the current R&D expenditures $Z_i(t)$ in this sector.
- Demand equations using equilibrium prices:

$$\Omega_i(t) \propto \left(\frac{\chi\, C(t)^{\epsilon_i}}{E(t)\; K_i(t)} \right)^{1-\sigma}, \qquad (37)$$

$$E(t) = \frac{1+\theta}{\theta} \frac{L^p(t)}{L(t)} = \left(\sum_{i=1}^{I} \left(\chi K_i(t)^{-1}\, C(t)^{\epsilon_i} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \qquad (38)$$

Dynamics Knowledge Stocks III*

- Aggregate allocation of employment in the R&D sector as a function $(C(t), \boldsymbol{K}(t))$, as

$$
\begin{aligned}
Z(t) =& L(t) - L^{p}(t) \\
=& L(t) \cdot \left[1 - \frac{\theta}{1+\theta} \left(\sum_{i=1}^{I} \left(\chi K_i(t)^{-1} C(t)^{\epsilon_i} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \right].
\end{aligned}
$$

- Imposes the following constraint along equilibrium path

$$
\sum_i B_i(t)^{\frac{1}{\alpha}} A_i(t)^{\frac{1}{\alpha}} = L(t) \cdot \left[1 - \frac{\theta}{1+\theta} E(t) \right].
$$

- Thus, dynamics expressed in terms of $(C(t), \boldsymbol{K}(t))$ alone.
- Observation: $\{\boldsymbol{\Omega}(t), \boldsymbol{\Lambda}(t)\}$ depend only on $(C(t), \boldsymbol{K}(t))$.
  - where $\Lambda_i(t) \equiv A_i(t) / \sum_{i'} A_{i'}(t)$.
- Back