1 Gross and Value Added Production functions

Gross value added:

$$\tilde{Y}_j = \tilde{A}_j \left[k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j} \right]^{1-\nu_j} x_j^{\nu_j}$$

$$x_j = \left[\sum_i \left(\omega_{ij}^X\right)^{\frac{1}{\sigma^X}} \left(x_{ij}\right)^{1-\frac{1}{\sigma^X}} \right]^{\frac{\sigma^X}{\sigma^X-1}}$$

Accounting identity (real terms)

$$\tilde{Y}_j = \sum_i x_{ji} + C_j + I_j$$

Demand of x_j

$$x_j = \nu_j \frac{\tilde{P}_j}{\tilde{P}_x} \tilde{Y}_j$$

Demand of x_{ij}

$$x_{ij} = \omega_{ij}^X \left(\frac{\tilde{P}_i}{\tilde{P}_x} \right)^{-\sigma^X} x_j$$

Combining these expressions:

$$x_{ij} = \omega_{ij}^X \left(\frac{\tilde{P}_i}{\tilde{P}_x} \right)^{-\sigma^X} \nu_j \frac{\tilde{P}_j}{\tilde{P}_x} \tilde{Y}_j$$

Subsitute in the accounting identity for output of sector j:

$$\tilde{Y}_j = \sum_i \nu_i \omega_{ji}^X \left(\frac{\tilde{P}_j}{\tilde{P}_x} \right)^{-\sigma^X} \frac{\tilde{P}_i}{\tilde{P}_x} \tilde{Y}_i + C_j + I_j$$

Simplifying example without investment

$$\tilde{Y}_j = \sum_i \nu_i \omega_{ji}^X \left(\frac{\tilde{P}_j}{\tilde{P}_x} \right)^{-\sigma^X} \frac{\tilde{P}_i}{\tilde{P}_x} \tilde{Y}_i + C_j$$

Consumption aggregator

$$C = \left[\sum_j \left(\omega_j^C\right)^{\frac{1}{\sigma^C}} C_j^{1-\frac{1}{\sigma^C}} \right]^{\frac{\sigma^C}{\sigma^C-1}}$$

2 Solving the simplified model

Aggregation of intermediates for sector j :

$$\max_{(x_{ij})_i} P_{xj} \left[\sum_i \left(\omega_i^X\right)^{\frac{1}{\sigma^X}} (x_{ij})^{1-\frac{1}{\sigma^X}} \right]^{\frac{\sigma^X}{\sigma^X-1}} - \sum_i \tilde{P}_i x_{ij}$$

FOCs:

$$P_{xj} \left[\sum_i \left(\omega_i^X\right)^{\frac{1}{\sigma^X}} (x_{ij})^{1-\frac{1}{\sigma^X}} \right]^{\frac{\sigma^X}{\sigma^X-1}-1} \left(\omega_i^X\right)^{\frac{1}{\sigma^X}} (x_{ij})^{-\frac{1}{\sigma^X}} = \tilde{P}_i$$

$$P_{xj} \left[\sum_i \left(\omega_{ij}^X\right)^{\frac{1}{\sigma^X}} (x_{ij})^{1-\frac{1}{\sigma^X}} \right]^{\frac{1}{\sigma^X-1}} \left(\omega_i^X\right)^{\frac{1}{\sigma^X}} (x_{ij})^{-\frac{1}{\sigma^X}} = \tilde{P}_i$$

$$P_{xj} x_j^{\frac{1}{\sigma^X}} \left(\omega_{ij}^X\right)^{\frac{1}{\sigma^X}} (x_{ij})^{-\frac{1}{\sigma^X}} = \tilde{P}_i$$

$$x_{ij} = \omega_{ij}^X \left(\frac{P_{xj}}{\tilde{P}_i}\right)^{\sigma^X} x_j$$

$$P_{xj} = \left[\sum_i \omega_{ij}^X \left(\tilde{P}_i\right)^{1-\sigma^X} \right]^{\frac{1}{1-\sigma^X}}$$

Producer's problem sector j :

$$\max_{x_j} \tilde{P}_j \tilde{A}_j \left(k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j}\right)^{1-\nu_j} x_j^{\nu_j} - P_{xj} x_j,$$

$$\nu_j \tilde{P}_j \tilde{A}_j \left(k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j}\right)^{1-\nu_j} x_j^{\nu_j-1} = P_{xj}$$

$$x_j = \left(\nu_j \frac{\tilde{P}_j}{P_{xj}} \tilde{A}_j\right)^{\frac{1}{1-\nu_j}} k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j}$$

Define value-added production as the problem where we have solved-out intermediates. We start from the accounting definition of nominal value added being equal to nominal gross output minus intermediate expenditures and rearrange

$$\begin{aligned}
P_j Y_j &= \tilde{P}_j \tilde{A}_j \left(\nu_j \frac{\tilde{P}_j}{P_{xj}} \tilde{A}_j\right)^{\frac{\nu_j}{1-\nu_j}} k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j} - P_{xj} \left(\nu_j \frac{\tilde{P}_j}{P_{xj}} \tilde{A}_j\right)^{\frac{1}{1-\nu_j}} k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j} \\
&= \tilde{P}_j \tilde{A}_j \left(\nu_j \frac{\tilde{P}_j}{P_{xj}} \tilde{A}_j\right)^{\frac{\nu_j}{1-\nu_j}} (1-\nu_j) k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j}
\end{aligned}$$

We want to obtain an expression for "real" value added:

$$Y_j = A_j k_j^{\alpha_j} l_j^{\eta_j} n_j^{1-\alpha_j-\eta_j}$$

We start from the nominal expression and check which parts need to be equal in the VA value added representation and definition:

$$\begin{aligned}
P_j A_j &= \tilde{P}_j \tilde{A}_j \left(\nu_j \frac{\tilde{P}_j}{\left[\sum_i \omega_{ij}^X \left(\tilde{P}_i\right)^{1-\sigma^X}\right]^{\frac{1}{1-\sigma^X}}} \tilde{A}_j \right)^{\frac{\nu_j}{1-\nu_j}} (1-\nu_j) \\
&= \left(\tilde{P}_j \tilde{A}_j\right)^{\frac{1}{1-\nu_j}} \left(\frac{\nu_j}{\left[\sum_i \omega_i^X \left(\tilde{P}_i\right)^{1-\sigma^X}\right]^{\frac{1}{1-\sigma^X}}} \right)^{\frac{\nu_j}{1-\nu_j}} (1-\nu_j) \\
\left(\tilde{P}_j \tilde{A}_j\right)^{\frac{1}{1-\nu_j}} &= \frac{P_j A_j}{1-\nu_j} \left(\frac{\left[\sum_i \omega_i^X \left(\tilde{P}_i\right)^{1-\sigma^X}\right]^{\frac{1}{1-\sigma^X}}}{\nu_j} \right)^{\frac{\nu_j}{1-\nu_j}}
\end{aligned}$$

Finally, think about a firm combining intermediates and VA to produce gross output. Solving the problem of a firm combining value added and the intermediate aggregate of sector j we obtain the price deflator for VA

$$\max \tilde{P}_j \tilde{A}_j y_j^{1-\nu_j} x_j^{\nu_j} - P_j y_j - P_{xj} x_j$$

$$(1-\nu_j) \tilde{P}_j \tilde{A}_j y_j^{-\nu_j} x_j^{\nu_j} = P_j$$

$$\nu_j \tilde{P}_j \tilde{A}_j y_j^{1-\nu_j} x_j^{\nu_j - 1} = P_{xj}$$

$$\frac{1-\nu_j}{\nu_j} \frac{x_j}{y_j} = \frac{P_j}{P_{xj}}$$

$$(1-\nu_j) \tilde{P}_j \tilde{A}_j \left(\frac{\nu_j}{1-\nu_j} \frac{P_j}{P_{xj}} \right)^{\nu_j} = P_j$$

$$\tilde{P}_j = \frac{1}{\tilde{A}_j} \left(\frac{P_j}{1-\nu_j} \right)^{1-\nu_j} \left(\frac{P_{xj}}{\nu_j} \right)^{\nu_j}$$

$$\tilde{P}_j = \frac{1}{\tilde{A}_j} \left(\frac{P_j}{1-\nu_j} \right)^{1-\nu_j} \left(\frac{\left[\sum_i \omega_{ij}^X \left(\tilde{P}_i\right)^{1-\sigma^X}\right]^{\frac{1}{1-\sigma^X}}}{\nu_j} \right)^{\nu_j}$$

[there may be some typos, inconsistencies, please point them out if you find them]